Eigenvalues of Disjointness Graphs

CombinaTexas 2024

Nathan Lindzey
Technion – Israel Institute of Technology

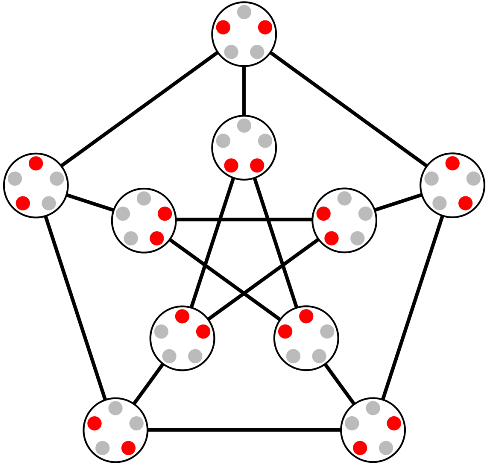
March 24, 2024

The Kneser Graph

Let $KG_{n,k}$ be the graph on $\binom{[n]}{k}$ such that $S \sim T \Leftrightarrow S \cap T = \emptyset$ for all $S, T \in \binom{[n]}{k}$.

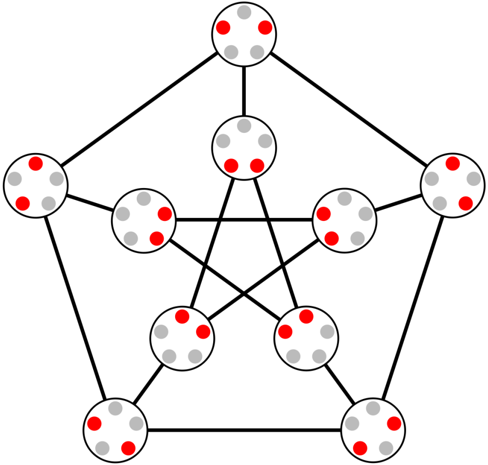
The Kneser Graph

Let $KG_{n,k}$ be the graph on $\binom{[n]}{k}$ such that $S \sim T \Leftrightarrow S \cap T = \emptyset$ for all $S, T \in \binom{[n]}{k}$.



The Kneser Graph

Let $KG_{n,k}$ be the graph on $\binom{[n]}{k}$ such that $S \sim T \Leftrightarrow S \cap T = \emptyset$ for all $S, T \in \binom{[n]}{k}$.



The Kneser Graph

In 1979, Lovász gave a spectral proof of the Erdős–Ko–Rado Theorem.

Theorem (The Erdős–Ko–Rado Theorem)

Let $n \geq 2k$ *and* $\mathcal{F} \subseteq \binom{[n]}{k}$ *be an intersecting family, i.e,* $S \cap T \neq \emptyset \ \forall S, T \in \mathcal{F}$. *Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

The Kneser Graph

In 1979, Lovász gave a spectral proof of the Erdős–Ko–Rado Theorem.

Theorem (The Erdős–Ko–Rado Theorem)

Let $n \geq 2k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family, i.e, $S \cap T \neq \emptyset\ \forall S, T \in \mathcal{F}$. Then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Equivalently, $\alpha(KG_{n,k}) = \binom{n-1}{k-1}$.

Eigenvalues and The Hoffman Bound

Theorem (Hoffman 70's)

Let $S \subseteq V$ be an independent set of a d-regular graph $G = (V, E)$. Let $\lambda_{\min}$ be the least eigenvalue of its adjacency matrix. Then

$$|S| \leq |V| \frac{-\lambda_{\min}}{d - \lambda_{\min}}.$$

Eigenvalues and The Hoffman Bound

Theorem (Hoffman 70's)

Let $S \subseteq V$ be an independent set of a d-regular graph $G = (V, E)$. Let $\lambda_{\min}$ be the least eigenvalue of its adjacency matrix. Then

$$|S| \leq |V| \frac{-\lambda_{\min}}{d - \lambda_{\min}}.$$

Theorem (Lovász '79)

The eigenvalues of $KG_{n,k}$ are $\lambda_j = (-1)^j \binom{n-k-j}{k-j}$ for all $j = 0, 1, \ldots, k$.

$$\binom{n-1}{k-1} \leq |S|$$

Eigenvalues and The Hoffman Bound

Theorem (Hoffman 70's)

Let $S \subseteq V$ be an independent set of a d-regular graph $G = (V, E)$. Let $\lambda_{\min}$ be the least eigenvalue of its adjacency matrix. Then

$$|S| \leq |V| \frac{-\lambda_{\min}}{d - \lambda_{\min}}.$$

Theorem (Lovász '79)

The eigenvalues of $KG_{n,k}$ are $\lambda_j = (-1)^j \binom{n-k-j}{k-j}$ for all $j = 0, 1, \ldots, k$.

$$\binom{n-1}{k-1} \leq |S| \leq \binom{n}{k} \frac{-\binom{n-k-1}{k-1}}{\binom{n-k}{k} - \binom{n-k-1}{k-1}}$$

Eigenvalues and The Hoffman Bound

Theorem (Hoffman 70's)

Let $S \subseteq V$ be an independent set of a d-regular graph $G = (V, E)$. Let $\lambda_{\min}$ be the least eigenvalue of its adjacency matrix. Then

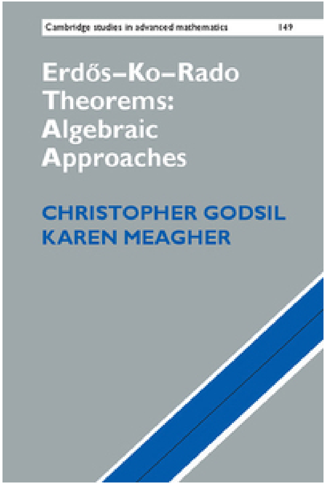
$$|S| \leq |V| \frac{-\lambda_{\min}}{d - \lambda_{\min}}.$$

Theorem (Lovász '79)

The eigenvalues of $KG_{n,k}$ are $\lambda_j = (-1)^j \binom{n-k-j}{k-j}$ for all $j = 0, 1, \ldots, k$.

$$\binom{n-1}{k-1} \leq |S| \leq \binom{n}{k} \frac{-\binom{n-k-1}{k-1}}{\binom{n-k}{k} - \binom{n-k-1}{k-1}} = \binom{n-1}{k-1} \quad \square$$

Erdős-Ko-Rado Combinatorics



- Sets
- Words
- Groups
- Matrices
- Partitions
- Permutations
- Vector Spaces
- Finite Geometries
- Perfect Matchings
- ⋮

Disjointness Graphs

A poset $(X, \leq)$ is a *meet-semilattice* if $(x \wedge y) \in X$ for any pair $x, y \in X$.

Disjointness Graphs

A poset $(X, \leq)$ is a *meet-semilattice* if $(x \wedge y) \in X$ for any pair $x, y \in X$.

Two elements $x, y \in X$ are *disjoint* if $x \wedge y = \emptyset$.

Disjointness Graphs

A poset $(X, \leq)$ is a *meet-semilattice* if $(x \wedge y) \in X$ for any pair $x, y \in X$.

Two elements $x, y \in X$ are *disjoint* if $x \wedge y = \emptyset$.

Let X^* be the set of maximal elements of a finite meet-semilattice $(X, \leq)$.

Disjointness Graphs

A poset $(X, \leq)$ is a *meet-semilattice* if $(x \wedge y) \in X$ for any pair $x, y \in X$.

Two elements $x, y \in X$ are *disjoint* if $x \wedge y = \emptyset$.

Let X^* be the set of maximal elements of a finite meet-semilattice $(X, \leq)$.

Let Φ be the *disjointness graph* of $(X, \leq)$, i.e., $x \sim y$ if $x \wedge y = \emptyset$ for all $x, y \in X^*$.

Disjointness Graphs

A poset $(X, \leq)$ is a *meet-semilattice* if $(x \wedge y) \in X$ for any pair $x, y \in X$.

Two elements $x, y \in X$ are *disjoint* if $x \wedge y = \emptyset$.

Let X^* be the set of maximal elements of a finite meet-semilattice $(X, \leq)$.

Let Φ be the *disjointness graph* of $(X, \leq)$, i.e., $x \sim y$ if $x \wedge y = \emptyset$ for all $x, y \in X^*$.

For each $x \in X$, let v_x be the vector indexed by maximal elements $x^* \in X^*$ such that

$$v_x(x^*) = \begin{cases} 1 & \text{if } x \leq x^*; \\ 0 & \text{otherwise.} \end{cases}$$

Disjointness Graphs

A poset $(X, \leq)$ is a *meet-semilattice* if $(x \wedge y) \in X$ for any pair $x, y \in X$.

Two elements $x, y \in X$ are *disjoint* if $x \wedge y = \emptyset$.

Let X^* be the set of maximal elements of a finite meet-semilattice $(X, \leq)$.

Let Φ be the *disjointness graph* of $(X, \leq)$, i.e., $x \sim y$ if $x \wedge y = \emptyset$ for all $x, y \in X^*$.

For each $x \in X$, let v_x be the vector indexed by maximal elements $x^* \in X^*$ such that

$$v_x(x^*) = \begin{cases} 1 & \text{if } x \leq x^*; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

If $(X, \leq)$ is a meet-semilattice, then $\Phi = \sum_{x \in X} \mu(\emptyset, x)\ v_x v_x^\top$.

Regularity Assumptions

Natural semilattices often

- are *graded*: $X = \{\emptyset\} \sqcup X_1 \sqcup \cdots \sqcup X_{n-1} \sqcup X_n^*$,

Regularity Assumptions

Natural semilattices often

- are *graded*: $X = \{\emptyset\} \sqcup X_1 \sqcup \cdots \sqcup X_{n-1} \sqcup X_n^*$,
- admit natural *group actions*,

Regularity Assumptions

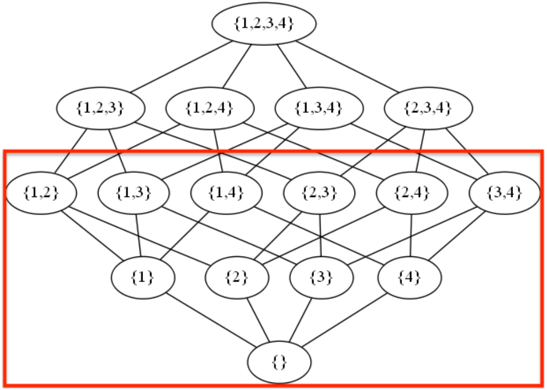
Natural semilattices often

- are *graded*: $X = \{\emptyset\} \sqcup X_1 \sqcup \cdots \sqcup X_{n-1} \sqcup X_n^*$,
- admit natural *group actions*,
- and have invariant Möbius functs: $\mu_k \equiv \mu(\emptyset, x) = \mu(\emptyset, x') \; \forall x, x' \in X_k \; 0 \leq k \leq n$.

Regularity Assumptions

Natural semilattices often

- are *graded*: $X = \{\emptyset\} \sqcup X_1 \sqcup \cdots \sqcup X_{n-1} \sqcup X_n^*$,
- admit natural *group actions*,
- and have invariant Möbius functs: $\mu_k \equiv \mu(\emptyset, x) = \mu(\emptyset, x') \ \forall x, x' \in X_k \ 0 \leq k \leq n$.



Zeta Decomposition

Define $Z_k = \sum_{x \in X_k} v_x v_x^\top$ as the *k-th zeta matrix* of a graded meet-semilattice $(X, \leq)$.

Zeta Decomposition

Define $Z_k = \sum_{x \in X_k} v_x v_x^\top$ as the *k-th zeta matrix* of a graded meet-semilattice $(X, \leq)$.

Theorem (Zeta Decomposition)

Let $(X, \leq)$ be a graded meet-semilattice such that μ is rank-invariant. Then the disjointness graph Φ of $(X, \leq)$ admits the following decomposition:

$$\Phi = \sum_{k=0}^{n} \mu_k Z_k.$$

Zeta Decomposition

Define $Z_k = \sum_{x \in X_k} v_x v_x^\top$ as the *k-th zeta matrix* of a graded meet-semilattice $(X, \leq)$.

Theorem (Zeta Decomposition)

Let $(X, \leq)$ be a graded meet-semilattice such that μ is rank-invariant. Then the disjointness graph Φ of $(X, \leq)$ admits the following decomposition:

$$\Phi = \sum_{k=0}^{n} \mu_k Z_k.$$

- The Z_k's have non-negative eigenvalues that are easier to compute.

Zeta Decomposition

Define $Z_k = \sum_{x \in X_k} v_x v_x^\top$ as the *k-th zeta matrix* of a graded meet-semilattice $(X, \leq)$.

Theorem (Zeta Decomposition)

Let $(X, \leq)$ be a graded meet-semilattice such that μ is rank-invariant. Then the disjointness graph Φ of $(X, \leq)$ admits the following decomposition:

$$\Phi = \sum_{k=0}^{n} \mu_k Z_k.$$

- The Z_k's have non-negative eigenvalues that are easier to compute.
- For sufficiently regular $(X, \leq)$ the zeta matrices commute ...

Zeta Decomposition

Define $Z_k = \sum_{x \in X_k} v_x v_x^\top$ as the *k-th zeta matrix* of a graded meet-semilattice $(X, \leq)$.

Theorem (Zeta Decomposition)

Let $(X, \leq)$ be a graded meet-semilattice such that μ is rank-invariant. Then the disjointness graph Φ of $(X, \leq)$ admits the following decomposition:

$$\Phi = \sum_{k=0}^{n} \mu_k Z_k.$$

- The Z_k's have non-negative eigenvalues that are easier to compute.
- For sufficiently regular $(X, \leq)$ the zeta matrices commute ...
- ... eigenvalues of Φ are then an alternating sum of the eigenvalues of the Z_k's.

Permutations

Let $X_n^* = S_n$ be the group of permutations of $[n]$.

Permutations

Let $X_n^* = S_n$ be the group of permutations of $[n]$.

X_n^* is the set of maximal elements of the $(n \times n)$-*chessboard complex* $(X, \leq)$ where

Permutations

Let $X_n^* = S_n$ be the group of permutations of $[n]$.

X_n^* is the set of maximal elements of the $(n \times n)$-*chessboard complex* $(X, \leq)$ where

$$X_k = \left\{ \{(i_1, j_1), \ldots, (i_k, j_k)\} \in \binom{[n] \times [n]}{k} : i_1, \ldots, i_k \text{ distinct}, j_1, \ldots, j_k \text{ distinct} \right\}.$$

Permutations

Let $X_n^* = S_n$ be the group of permutations of $[n]$.

X_n^* is the set of maximal elements of the $(n \times n)$-*chessboard complex* $(X, \leq)$ where

$$X_k = \left\{ \{(i_1, j_1), \ldots, (i_k, j_k)\} \in \binom{[n] \times [n]}{k} : i_1, \ldots, i_k \text{ distinct}, j_1, \ldots, j_k \text{ distinct} \right\}.$$

Its disjointness graph Φ_n is known as *the derangement graph* $\mathcal{D}_n = \mathrm{Cay}(S_n, D_n)$.

Permutations

Let $X_n^* = S_n$ be the group of permutations of $[n]$.

X_n^* is the set of maximal elements of the $(n \times n)$-*chessboard complex* $(X, \leq)$ where

$$X_k = \left\{ \{(i_1, j_1), \ldots, (i_k, j_k)\} \in \binom{[n] \times [n]}{k} : i_1, \ldots, i_k \text{ distinct}, j_1, \ldots, j_k \text{ distinct} \right\}.$$

Its disjointness graph Φ_n is known as *the derangement graph* $\mathcal{D}_n = \mathrm{Cay}(S_n, D_n)$.

Theorem (Deza–Frankl '79, Cameron–Ku '03, Larose–Malvenuto '04, Renteln '07, Ellis, Friedgut–Pilpel '08, Godsil–Meagher '09)

For all $n \geq 2$, we have $\alpha(\mathcal{D}_n) = (n-1)!$.

The Derangement Graph

The eigenvalues η_λ of $\mathcal{D}_n$ are much more difficult to compute than the Kneser graph.

The Derangement Graph

The eigenvalues η_λ of $\mathcal{D}_n$ are much more difficult to compute than the Kneser graph.

$$\eta_\lambda = \frac{1}{\chi^\lambda(1)} \sum_{\pi \in D_n} \chi^\lambda(\pi).$$

Determinantal and recursive expressions for η_λ were determined by Renteln '07.

The Derangement Graph

The eigenvalues η_λ of $\mathcal{D}_n$ are much more difficult to compute than the Kneser graph.

$$\eta_\lambda = \frac{1}{\chi^\lambda(1)} \sum_{\pi \in D_n} \chi^\lambda(\pi).$$

Determinantal and recursive expressions for η_λ were determined by Renteln '07.

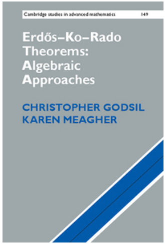
Since $\mathcal{D}_n$ is the disjointness graph Φ_n of the $(n \times n)$-chessboard complex, we have

$$\mathcal{D}_n = \Phi_n = \sum_{k=0}^{n} \mu_k Z_k = \sum_{k=0}^{n} (-1)^k Z_k.$$

The zeta matrices Z_k commute.

Eigenvalues of the Derangement Graph

Open Question: Lovász-type result (closed forms) for the eigenvalues of $\mathcal{D}_n$?



the eigenvalues of the perfect matching graph $M(2k)$ for $k \leq 5$. Ideally, we would like to have a closed form for all the eigenvalues of all the perfect matching graphs, but this seems to be a very difficult problem. Toward this

Theorem (L. '23)

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$ and η_λ be the λ-eigenvalue of $\mathcal{D}_n$. Then

$$\eta_\lambda = (-1)^{n-\lambda_1} D_\lambda$$

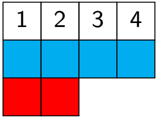
where D_λ is the number of λ-colored derangements.

λ-Colored Permutations and Derangements

Let $\lambda = (4, 4, 2) \vdash 10$.

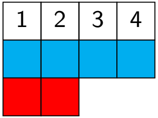
λ-Colored Permutations and Derangements

Let $\lambda = (4, 4, 2) \vdash 10$.



λ-Colored Permutations and Derangements

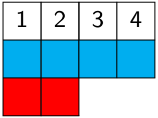
Let $\lambda = (4, 4, 2) \vdash 10$.



Take $\sigma \in \text{Sym}([\lambda_1])$ and assign colors to symbols so that cycles are *monochromatic*.

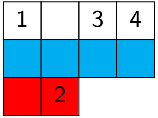
λ-Colored Permutations and Derangements

Let $\lambda = (4, 4, 2) \vdash 10$.

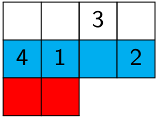


Take $\sigma \in \mathrm{Sym}([\lambda_1])$ and assign colors to symbols so that cycles are *monochromatic*.

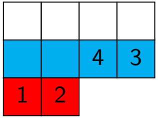
$(1)(2)(3)(4)$



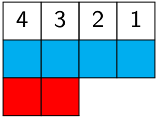
$(1, 2, 4)(3)$



$(1)(2)(3, 4)$

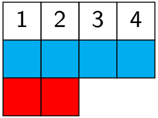


$(1, 4)(2, 3)$



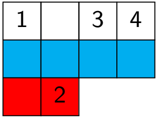
λ-Colored Permutations and Derangements

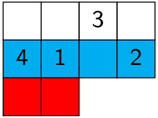
Let $\lambda = (4, 4, 2) \vdash 10$.

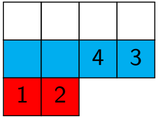


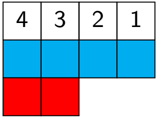
Take $\sigma \in \mathrm{Sym}([\lambda_1])$ and assign colors to symbols so that cycles are *monochromatic*.

$(1)(2)(3)(4)$ $\quad$ $(1, 2, 4)(3)$ $\quad$ $(1)(2)(3, 4)$ $\quad$ $(1, 4)(2, 3)$









$D_\lambda := \#$ λ-colored permutations such that no white symbol gets sent to itself.

Closed Forms?

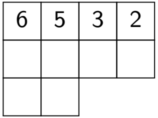
$\#$ λ-colored permutations $= h_\lambda(1,1)h_\lambda(1,2)\cdots h_\lambda(1,\lambda_1) =: H^1(\lambda)$.

6	5	3	2

$= 180$

Closed Forms?

$\#$ λ-colored permutations $= h_\lambda(1,1)h_\lambda(1,2)\cdots h_\lambda(1,\lambda_1) =: H^1(\lambda)$.

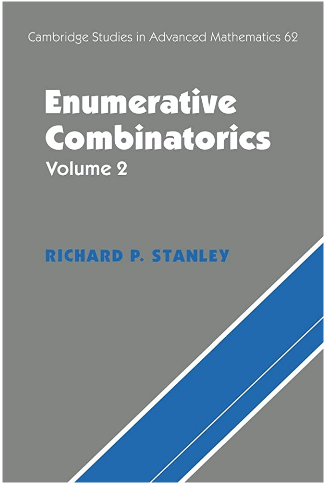


$= 180$

$\#$ λ-colored derangements $D_\lambda \approx H^1(\lambda)e_\lambda^{-1}$

(L. '23) Explicit closed form for D_λ derived via the calculus of finite differences.

Character Sums of Derangements



7.63. **a.** [2+] For $\lambda \vdash n$ define

$$d_\lambda = \sum_{w \in \mathfrak{D}_n} \chi^\lambda(w),$$

where $\mathfrak{D}_n$ denotes the set of all derangements (permutations without fixed points) in $\mathfrak{S}_n$. Show that

$$\sum_{\lambda \vdash n} d_\lambda s_\lambda = \sum_{k=0}^{n} (-1)^{n-k} (n)_k h_1^{n-k} h_k.$$

b. [2+] Deduce from (a) that for $1 \le k \le n$,

$$d_{\langle j, 1^{n-j}\rangle} = (-1)^{n-j} \binom{n}{j} D_j + (-1)^{n-1} \binom{n-1}{j},$$

where $D_j = \#\mathfrak{D}_j$ (discussed in Example 2.2.1).

Immanants of the Complete Graph K_n

Let $\lambda \vdash n$ be an integer partition.

The λ-*immanant* of a $n \times n$ matrix A is defined such that

$$\mathrm{Imm}_\lambda(A) = \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\, A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)}.$$

For $\chi^{(1^n)} = \mathrm{sgn}$, we recover the *determinant*. For $\chi^{(n)} = 1$, we recover the *permanent*.

(L. '23) Closed form expressions for $\mathrm{Imm}_\lambda(J_n - I_n)$ where J_n is the all-ones matrix:

$$\mathrm{Imm}_\lambda(J_n - I_n) = \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\, (1 - \delta_{1,\sigma(1)}) \cdots (1 - \delta_{n,\sigma(n)}) = \sum_{\sigma \in D_n} \chi^\lambda(\sigma) = d_\lambda.$$

Other combinatorial/binary matrices with 'nice' immanants?

Immanants of the Complete Graph K_n

Let $\lambda \vdash n$ be an integer partition.

The λ-*immanant* of a $n \times n$ matrix A is defined such that

$$\mathrm{Imm}_\lambda(A) = \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\ A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{n,\sigma(n)}.$$

For $\chi^{(1^n)} = \mathrm{sgn}$, we recover the *determinant*. For $\chi^{(n)} = 1$, we recover the *permanent*.

(L. '23) Closed form expressions for $\mathrm{Imm}_\lambda(J_n - I_n)$ where J_n is the all-ones matrix:

$$\mathrm{Imm}_\lambda(J_n - I_n) = \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\ (1 - \delta_{1,\sigma(1)}) \cdots (1 - \delta_{n,\sigma(n)}) = \sum_{\sigma \in D_n} \chi^\lambda(\sigma) = d_\lambda.$$

Other combinatorial/binary matrices with 'nice' immanants?

That's all. Thanks!